ON THE TRANSVERSE BENDING OF A PLATE SUPPORTED ALONG THE EDGES AND CONSISTING OF SEVERAL CLOSED CURVES

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1. We assume that the middle surface of the bent plate lies in the plane of the complex variable $z = x + iy$ in a finite multiply-connected region S, bounded by a totality of sufficiently smooth closed curves $L_j (j = 1, \ldots, m + 1)$ without common points; of these curves, let L_{m+1} be the external contour surrounding the internal boundaries $L_j (j = 1, \ldots, m)$. The entire contour of the region S is denoted by L, formed by the curves $L_j (j = 1, \ldots, m + 1)$; its circuit is made in a positive direction relative to S. Further, by S_j (finite for $j \neq m + 1$ and infinite for $j = m + 1$) we denote a simply-connected region bounded by L_j $(j = 1, \ldots, m + 1)$. We denote a finite and simply-connected region bounded by the external contour L_{m+1} by S_0. Let z_j be an arbitrary fixed point lying in the region $S_j (j = 1, \ldots, m)$, and a_j be an affix of a certain point on the curve L_j $(j = 1, \ldots, m + 1)$ selected as an origin of arc. For convenience we take as the origin of coordinates a point in S.

The unknown deflection $w_1(x, y)$ of the middle surface of the plate may be written in the form

$$w_1(x, y) = w(x, y) + w_0(x, y) \tag{1.1}$$

where $w_0(x, y)$ is some particular solution of the differential equation of bending, describing the effect of a normal force distributed according to some specified law; the new unknown $w(x, y)$ is a biharmonic function represented in accordance with the Goursat formula in the form

$$2w(x, y) = \bar{z}\varphi_1(z) + z\overline{\varphi_1(z)} + \chi_1(z) + \overline{\chi_1(z)}, \quad \psi_1(z) = \chi_1'(z) \tag{1.2}$$

where $\phi_1(z)$ and $\psi_1(z)$ are certain analytic and generally multiple-valued functions in the multiply-connected region S. For the case in question, that of a supported edge of the plate, we shall have the following limiting equation for determining the functions $\phi_1(z)$ and $\psi_1(z)$ on the contour L:

$$\operatorname{Re} t^{\cdot} [\overline{\varphi_1(t)} + \bar{t}\varphi_1'(t) + \psi_1(t)] = f_1(s) \qquad (1.3)$$

$$\operatorname{Re} \{(1+\lambda_0)\varphi_1'(t) + \overline{\varphi_1'(t)} + t^{\cdot\cdot} [\overline{\varphi_1(t)} + \bar{t}\varphi_1'(t) + \psi_1(t)]\} = f_2(s) \qquad (1.4)$$

in which by the variable t is understood the complex coordinate of a running point on L, and the dots beside it denote differentiation with respect to the arc s; λ_0 is a constant depending upon Poisson's ratio; and finally, $f_1(s)$ and $f_2(s)$ are certain given functions determined by the nature of the loading, which we shall consider to be continuous on L for the case of plate bending.

Note 1. Khalilov [1] has reduced the case of bending of a supported plate to a Fredholm integral equation in the case where the middle surface occupies a simply-connected region bounded by a contour of non-zero curvature. For the more general case of a multiply-connected region without the above restriction on boundary curvature but different in some respects from the usual treatment, the problem of the supported plate, including its reduction to a Fredholm equation, has been considered by Fridman in a different way [2].

Kalandiia studied this same problem for a multiply-connected region in the usual way and reduced it to a singular integral equation; he has established its solvability [3] on the basis of comparatively recent methods for the investigation of such equations [4].

In the first two articles, the Fredholm integral equation was presented in a structurally complex form; the kernel was given by certain quadratures not as a rule expressible by elementary functions, a fact which naturally limits the application of the equations. The Fredholm equation which we mean to propose for our problem is free from this objection. Its kernel is expressed by elementary functions; further, there is an additional property inherent in the equation which facilitates its practical application. By making use of modern computational techniques it would not be difficult, on the basis of the proposed equation, to arrive at a satisfactory evaluation of effects in a quantitative manner. Certain important specific problems may be studied with the help of this equation by reducing them to quasiregular infinite systems of equations. It seems to us that the present paper must be considered from just this point of view. We note that the solution for finite and for simply-connected regions, a classic example treated in a similar style, has been reported in reference [5].

In place of $\phi_1(z)$ and $\psi_1(z)$ in equations (1.3) and (1.4) we introduce functions $\phi(z) = i\phi_1(z)$, $\psi(z) = i\psi_1(z)$ and set*

$$\varphi(z) = \sum_{j=1}^{m+1} \varphi_j(z), \qquad \psi(z) = \sum_{j=1}^{m=1} \psi_j(z) \tag{1.5}$$

where $\phi_{m+1}(z)$ and $\psi_{m+1}(z)$ are single-valued and regular in the region S_0; any two of the remaining functions $\phi_k(z)$ and $\psi_k(z)$ are analytic inside $L_k (k = 1, \ldots, m)$. To the left-hand side of boundary condition (1.3) we add the operator

$$\Gamma(\varphi, \psi; t) = \{(2S_0)^{-1} \operatorname{Re} it\bar{t}^{\cdot} \operatorname{Re} \varphi'_{m+1}(0), \ (w + w_0)_{t=a_{m+1}} - (w + w_0)_{t=a_j}\} \tag{1.6}$$

taking the first or the second of the expressions in the bracket according to whether point t lies on the outside contour L_{m+1} or on one of the inside contours $L_j (j \neq m + 1)$;** it is seen that on the internal boundary L_j the operator takes on a value equal to the difference of the values of $w(x, y)$ at the points $z = a_{m+1}$ and $z = a_j$. Both boundary conditions (1.3) and (1.4) are united into one limiting complex equation

$$\lambda[\varphi'(t) - \overline{\varphi'(t)}] + \overline{t^{\cdot}\theta_1(t)}\gamma(t) - t^{\cdot}\theta_2(t)\overline{\gamma(t)} + \Gamma(\varphi, \psi; t) = f(t) \tag{1.7}$$

in which the notation

$$\gamma(t) = \varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)}, \qquad \theta_1(t) = i + \bar{t}^{\cdot} t^{\cdot\cdot}, \qquad \theta_2(t) = -i + \bar{t}^{\cdot} t^{\cdot\cdot}$$

$$\lambda = 2 + \lambda_0 \qquad f(t) = 2(f_1 + if_2) \tag{1.8}$$

has been introduced.

The reader can convince himself that this modification of conditions (1.3) and (1.4) does not by any means involve any change in the postulation of the problem, as might be thought at first glance; the postulation not only remains unchanged in its original form but is appreciably simplified for the process of solution.

2. For the functions $\phi(z)$ and $\psi(z)$ we take the following forms, which will be justified later:

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* The presence of a number of simple functions with single subscripts on the right-hand portion of equation (1.5) must not lead the reader astray by his relating this notation to that for the unknown function in (1.2).

** If it is more convenient, the integral of $(w + w_0)$ taken along the arc of the curve $L_k (k = 1, \ldots, m + 1)$ may be substituted for $(w + w_0)_{t = a_k}$ in equation (1.6).

$$\varphi(z) = \varphi^{(0)}(z) + \sum_{j=1}^{m} iA_j(z - z_j)\ln(z - z_j)$$

$$A_j = \operatorname{Re}\frac{1}{2\pi i}\int_{L_j}\{\omega(t)\theta_1(t) + \overline{\omega(t)}\theta_2(t)\}\frac{dt}{t - z_j} \tag{2.1}$$

$$\psi(z) = \psi^{(0)}(z) - \sum_{j=1}^{m}\left\{(iA_j\bar{z}_j + 2\bar{B}_j)\left[\ln(z - z_j) + \frac{z_j}{z - z_j}\right] + (1 + i)\frac{D_j}{z - z_j}\right\} \tag{2.2}$$

where the functions $\phi^{(0)}(z)$ and $\psi^{(0)}(z)$, analytic in the region S, are given by the formulas

$$\varphi^{(0)}(z) = \int_L \{\omega(t)\theta_1(t) + \overline{\omega(t)}\theta_2(t)\} G(t, z)\,dt \tag{2.3}$$

$$\psi^{(0)}(z) = \int_L \{\omega(t) H(t, z) + \overline{\omega(t)} T(t, z)\}\,dt \tag{2.4}$$

Here $\omega(t)$ is the required density, and the functions introduced under the integral sign are such that

$$G(t, z) = \left\{\frac{1}{4\pi\lambda}\left[-1 + \ln\left(1 - \frac{z}{t}\right)\right], \qquad \frac{1}{4\pi\lambda}\ln(z - t)\right\} \tag{2.5}$$

$$H(t, z) = \left\{\bar{t}^{\cdot 2}\overline{\theta_2(t)}\left[G(t, z) + \frac{1}{4\pi\lambda}\varepsilon_j\right] + P(t)\left(\frac{1}{t - z} - \varepsilon_j\frac{1}{t}\right)\right\}$$

$$T(t, z) = \left\{\bar{t}^{\cdot 2}\overline{\theta_1(t)}\left[G(t, z) + \frac{1}{4\pi\lambda}\varepsilon_j\right] + Q(t)\left(\frac{1}{t - z} - \varepsilon_j\frac{1}{t}\right)\right\}$$

$$P(t) = \frac{1}{4\pi\lambda}(\lambda\bar{t}^{\cdot} - \overline{t\theta_1(t)}), \qquad Q(t) = \frac{1}{4\pi\lambda}(\lambda\bar{t}^{\cdot} - \overline{t\theta_2(t)}) \tag{2.6}$$

in which $\epsilon_j = 0$ for $j \neq m + 1$ and $\epsilon_{m+1} = 1$. The function $G(t, z)$, similar to that in (1.6), is equal to the first expression in (2.5) if t varies on L_{m+1} and to the second if t varies on the other $L_j (j = 1, \ldots, m)$. The quantities A_j, B_j and D_j are certain functions depending upon $\omega(t)$; the first was given in (2.1); the third is real, and for B_j and D_j we have

$$B_j = \frac{1}{4\pi\lambda}\int_{L_j}\{\omega(t)\theta_1(t) + \overline{\omega(t)}\theta_2(t)\}\,dt, \qquad D_j = \operatorname{Re}\int_{L_j}\varepsilon[\omega(t), t]\,ds \tag{2.7}$$

in which the notation

$$\varepsilon[\omega(t), t] = \frac{1}{2\pi}\left\{\frac{1}{\lambda}[\overline{tt^{\cdot}\theta_1(t)} - tt^{\cdot}\overline{\theta_2(t)}] - 1\right\}\omega(t) \tag{2.8}$$

has been introduced.

We make a cut in the region S connecting the point a_j on the internal boundary L_j with any point on the surrounding boundary L_{m+1} and not passing through the origin of coordinates. For this, the branch $\ln(1 - z/t)$ is taken in (2.5), which reduces to zero for $z = 0$; the branch $\ln(z - t)$ for the variable t associated with $L_j (j = 1, \ldots, m)$ is determined by fixing the argument of $\ln(z - a_j)$ for the affix of z found on either side of the cut connecting a_j with the curve L_{m+1}.

Certain other forms may be given to the functions $\phi(z)$ and $\psi(z)$ by virtue of (2.5) and (2.6), as follows:

$$\varphi(z) = \varphi^*(z) + \sum_{j=1}^{m} [iA_j(z - z_j) + B_j] \ln(z - z_j) \tag{2.9}$$

$$\psi(z) = \psi^*(z) - \sum_{j=1}^{m} \left\{(iA_j\bar{z}_j + \bar{B}_j)\left[\ln(z - z_j) + \frac{z_j}{z - z_j}\right] + i\,\frac{D_j}{z - z_j}\right. \tag{2.10}$$

In these equations $\phi^*(z)$ is a function which is regular in the region S; it is determined by the integral of (2.3) in which $G^*(t, z)$ must be substituted for the function $G(t, z)$; it takes on a value equal to $G(t, z)$ for an affix of t varying along L_{m+1} and for the t associated with $L_j (j = 1, \ldots, m)$ it has a value equal to

$$G^*(t, z) = \frac{1}{4\pi\lambda} \ln \frac{z - t}{z - z_j}$$

The function $\psi^*(z)$ and its integral are also regular in the region S. For this same function $\psi^*(z)$ may be expressed as in (2.4) if, in place of $H(t, z)$ and $T(t, z)$, the corresponding quantities $H^*(t, z)$ and $T^*(t, z)$ are substituted. For the latter, in turn, we may use formulas (2.6) by setting in them for $G(t, z) + (4\pi\lambda^{-1}$ and for $(t - z)^{-1} - \epsilon_j t^{-1}$ as the multiplier for the P and Q functions the following quantities respectively:

$$\begin{aligned} W^*(t, z) &= \left\{G(t, z) + \frac{1}{4\pi\lambda};\ \frac{1}{4\pi\lambda}\left(\ln\frac{z - t}{z - z_j} + \frac{t - z_j}{z - z_j}\right)\right\} \\ \sigma^*(t, z) &= \left\{\frac{1}{t - z} - \frac{1}{t};\ \frac{1}{t - z} + \frac{1}{z - z_j}\right\} \end{aligned} \tag{2.11}$$

It is clear that the functions $\phi_k(z)$ and $\psi_k(z)$ from (1.5) are determined partly from each of the equalities (2.1) and (2.2) or from (2.9) and (2.10), which contain the integrals taken along the contour L_k and also sums with additional operators for the index $j = k$. It is evident from the construction of (2.5) and (2.6) that it is necessary for

$$\psi_{m+1}(0) = 0 \tag{2.12}$$

It is expedient, also, to introduce the functions $\phi_k^{(0)}(z)$, $\psi_k^{(0)}(z)$ together with $\phi_k^*(z)$ and $\psi_k^*(z)$ $(k = 1, \ldots, m + 1)$; each of these functions is expressed by the integrals (2.3) and (2.4) and by analogous integrals taken along the curve L_k. It is not difficult to convince oneself that the function $\psi_k{}^*(z)$ is of the order $O[(z-z_k)^{-2}]$ at infinity.

We denote the expansions of the integrals of $\psi(z)$ and $\psi^*(z)$ in terms of z by $\chi(z)$ and $\chi^*(z)$, along an arbitrary path between the origin of coordinates and the point z without intersecting the cut and without leaving the region S. Simple calculations give

$$R^*(t,\ z)=\int_0^z W^*(t,\ z)\,dz, \qquad \Delta^*(t,\ z)=\int_0^z \sigma^*(t,\ z)\,dz$$

in which

$$R^*(t,\ z)=\left\{-\frac{1}{4\pi\lambda}\left[z+(t-z)\ln\left(1-\frac{z}{t}\right)\right],\ \frac{1}{4\pi\lambda}\left[(z-t)\ln\frac{z-t}{z-z_j}+t\ln\frac{t}{z_j}\right]\right\}$$

$$\Delta^*(t,\ z)=\left\{-\left[\ln\left(1-\frac{z}{t}\right)+\frac{z}{t}\right],\ -\left[\ln\frac{z-t}{z-z_j}-\ln\frac{t}{z_j}\right]\right\} \qquad (2.13)$$

The indicated choice of branch for the logarithmic functions above makes it compulsory to consider the right hand part of (2.13) as vanishing for $z=0$. The function $\chi^*(z)$ is formed from formulas similar to those for $\psi^*(z)$ if R^* and Δ^* are introduced into them in place of W^* and σ^*. In addition, it may readily be discovered that

$$\chi(z)=\chi^*(z)-\sum_{j=1}^{m}\Omega_j[\omega(t);\ z,\ z_j] \qquad (2.14)$$

in which, following the summation sign,

$$\Omega_j=(iA_j\bar{z}_j+\bar{B}_j)z[\ln(z-z_j)-1]+iD_j[\ln(z-z_j)-\ln(-z_j)] \qquad (2.15)$$

For $w(x,\ y)$, after recalling equations (1.2), (2.1) and (2.2) and their connection with the last two equations, we obtain the formula

$$2w(x,y)=2w^*(x,y)-\sum_{j=1}^{m}\delta_j{}^*(x,\ y;\ z_j) \qquad (2.16)$$

In this formula $w^*(x,\ y)$ is a biharmonic function equal to

$$2w^*(x,y)=-i\,\{\bar{z}\varphi^*(z)-z\overline{\varphi^*(z)}+\chi^*(z)-\overline{\chi^*(z)}\} \qquad (2.17)$$

and the value of the real operator $\delta_j{}^*(x,\ y;\ z_j)$ is given by the relation

$$\delta_j{}^*=A_j\delta_{1j}{}^*(x,y;z_j)+[B_j\delta_{2j}{}^*(x,y;z_j)+\bar{B}_j\overline{\delta_{2j}{}^*(x,y;z_j)}]+D_j\delta_{3j}{}^*(x,y;z_j) \quad (2.18)$$

in which, in turn

$$\begin{aligned} \delta_{1j}^* &= -\{[|z - z_j|^2 - |z_j|^2] \ln|z - z_j|^2 + (z_j\bar{z} + \bar{z}_j z)\} \\ \delta_{2j}^* &= i\bar{z}\,[\ln|z - z_j|^2 - 1], \qquad \delta_{3j}^* = \ln|z - z_j|^2 - \ln|z_j|^2 \end{aligned} \tag{2.19}$$

The integrals of equations (2.3), (2.4), (2.9) and (2.10), as well as the nature of the functions, permit each of the quantities $w(x, y)$ and $w^*(x, y)$ to be presented in identical form

$$[w(x, y);\; w^*(x, y)] = \int_L \{\omega(t)\,[Z(s;\; x, y);\; Z^*(s;\; x, y)] + \overline{\omega(t)}\,[\overline{Z(s;\; x, y)};\; \overline{Z^*(s; x, y)}]\}ds$$

where $Z(s;\; x,\; y)$ and $Z^*(s;\; x,\; y)$ are continuous functions of their variables. We remark that $w(x, y)$ and $w^*(x, y)$ consist of summations of $w_j(x, y)$ and $w_j^*(x, y)$ determined in a similar way, for example, to $\phi_j(z)$ and $\psi_j(z)$ in (1.5).

The operator (1.6) proves to be dependent on the density $\omega(t)$ just as do $\phi(z)$ and $\psi(z)$, and so it may be written as $\Gamma[\omega(t),\; t_0]$; we will adhere to this notation in the future.

Note 1. It is easy to understand that the operator $D_k/z - z_k$ has been introduced into the second summation in the right-hand part of (2.2) in order to remove multiple-valued terms which appear in some cases in the expression for $w(x, y)$; this can be discerned from formula (2.10). At the same time it may not be wholly clear at first glance what induced the author to attach the operator $iD_k/z - z_k$ to (2.2) while having introduced a single-valued sum in (2.15). The reason is that with this operator present it is easy in many cases (and even obligatory, please) to free the expansion of $\psi_k(z)$ in a circle at an infinitely remote point from a term in the sum having the inverse first power of $z - z_k$. Actually it is extremely likely that by fixing the value of $\omega(t)$ one may find a point z_k inside L_k such that the coefficient of the inverse first power term in the sum added to (2.10) reduces to zero for $j = k$. The presence of such a fact would indicate that the omission of the operator $iD_k/z - z_k$ from (2.2) imposes an excess of rigor and does not set correct limits on the function $\psi_k(z)$, and so leads to a loss of necessary generality in the expression for it. The introduction of the operator into (2.2) removes this defect at least formally, and, as we shall see below, does in fact do so. It is not without interest for the reader to consider also that the said defect in the representation of $\psi_k(z)$ may be completely taken care of by this very operator; we dwell on this because of the investigation of the Fredholm equation further on. We must suppose that there must exist many other operators apart from $iD_k/z - z_k$ capable of fulfilling this assignment and of giving the expression for $\psi_k(z)$ the required completeness. The operator we have chosen - whose very form partly prompts the nature of the premises leading to it - is the most natural, and at the same time certainly the simplest, of all the possible operators.

The exposition just set out bears an intuitive character and possesses no rigid basis; nevertheless it leads to an entirely correct idea, whose justification will later be corroborated by unquestionably substantial arguments having a direct relation to the problem in hand.

3. In equations (2.1), (2.2) and their first differentials, let z approach a certain point t_0 on the contour L. The limiting values of $\phi(t_0)$, $\phi'(t_0)$ and $\psi(t_0)$ are obtained from the boundary condition (1.7). Then, after performing certain calculations requiring a series of transformations, for the density $\omega(t)$ we obtain a Fredholm integral equation

$$\omega(t_0) + \int_L [\omega(t) M(t, t_0) + \overline{\omega(t)} N(t, t_0)]\, dt + O[\omega(t), t_0] = f(t_0) \quad (3.1)$$

where the kernels $M(t, t_0)$ and $N(t, t_0)$ are continuous functions of both t and t_0 and are respectively equal to

$$\begin{aligned} M(t, t_0) &= u(t, t_0) + h(t, t_0) \frac{d}{dt} \ln \frac{t - t_0}{\bar{t} - \bar{t}_0} + \frac{p(t, t_0)}{t - t_0} + q(t, t_0) \\ N(t, t_0) &= v(t, t_0) + l(t, t_0) \frac{d}{dt} \ln \frac{t - t_0}{\bar{t} - \bar{t}_0} + \frac{p(t, t_0)}{t - t_0} + r(t, t_0) \end{aligned} \quad (3.2)$$

Here the quantities on the right-hand side of these equations are found from the formulas

$$\begin{aligned} u(t, t_0) &= \bar{t}_0' \overline{\theta_1(t_0)}\, a(t, t_0) - t_0' \bar{t}'^2 \theta_2(t_0) \overline{b(t, t_0)} \\ v(t, t_0) &= \bar{t}_0' \overline{\theta_1(t_0)}\, b(t, t_0) - t_0' \bar{t}'^2 \theta_2(t_0) \overline{a(t, t_0)} \\ \frac{4\pi\lambda}{\theta_1(t)} a(t, z) &= \left\{-1 + \ln \frac{1 - (z/t)}{\overline{1 - (z/t)}},\ \ln \frac{z - t}{\overline{z - t}} + 2\, \overline{\ln(z - z_j)}\right\} \\ b(t, z) &= \frac{\theta_2(t)}{\theta_1(t)} a(t, z) \\ h(t, t_0) &= \frac{1}{4\pi} \{-\overline{\theta_2(t)} + \bar{t}_0' \overline{\theta_1(t_0)} [t' - d(t, t_0)] - t_0' \theta_2(t_0) \overline{c(t, t_0)}\} \\ l(t, t_0) &= \frac{1}{4\pi} \{-\overline{\theta_1(t)} + \bar{t}_0' \overline{\theta_1(t_0)} [t' - c(t, t_0)] - t_0' \theta_2(t_0) \overline{d(t, t_0)}\} \\ p(t, t_0) &= \frac{1}{4\pi} \{t' [\bar{t}' \overline{\theta_1(t)} - \bar{t}_0' \overline{\theta_1(t_0)}] - \bar{t}' [t' \theta_2(t) - t_0' \theta_2(t_0)]\} \\ c(t, t_0) &= \frac{1}{\lambda} (t - t_0) \theta_1(t), \qquad d(t, t_0) = \frac{1}{\lambda} (t - t_0) \theta_2(t) \\ q(t, t_0) &= \frac{1}{4\pi\lambda} [\bar{t}_0' \overline{\theta_1(t_0)}\, \theta_2(t) - t_0' \bar{t}'^2 \theta_2(t_0) \overline{\theta_1(t)}] \\ r(t, t_0) &= \frac{1}{4\pi\lambda} [\bar{t}_0' \overline{\theta_1(t_0)}\, \theta_1(t) - t_0' \bar{t}'^2 \theta_2(t_0) \overline{\theta_2(t)}] \end{aligned} \quad (3.3)$$

As is evident from (3.3), the function $p(t, t_0)$ reduces to zero for $t = t_0$; hence the third term in the sum of (3.2) will be a limiting function of t and t_0 on the assumption that the contour L has a differentiable curvature. Furthermore, the operator on the left-hand side of equation (3.2) is

$$O[\omega(t, t_0)] = \sum_{j=1}^{m} \{iA_j\alpha_j(t_0) + [\overline{\theta_1(t_0)}R_j(t_0) - \theta_2(t_0)\overline{R_j(t_0)}]\} + K[\omega(t), t_0] \tag{3.4}$$

in which new notation is used in the following sense:

$$\alpha_j(t_0) = \lambda[2 + \ln(t_0 - z_j) + \overline{\ln(t_0 - z_j)}] + \bar{t}_0^{\cdot}\overline{\theta_1(t_0)}\eta_j(t_0, z_j) + + t_0^{\cdot}\theta_2(t_0)\overline{\eta_j(t_0, z_j)}$$

$$K[\omega(t), t_0] = E\bar{t}_0^{\cdot}\overline{\theta_1(t_0)} - \bar{E}t_0^{\cdot}\theta_2(t_0) + \Gamma[\omega(t), t_0], \quad R_j(t_0) = F_j\frac{\bar{t}_0^{\cdot}}{\bar{t}_0 - \bar{z}_j}$$

$$\eta_j(z, z_j) = \left\{(z - z_j)[\ln(z - z_j) + \overline{\ln(z - z_j)}] - \frac{z_j\bar{z}_j}{\bar{z} - \bar{z}_j} + z\right\} \tag{3.5}$$

$$F_j = 2B_j\bar{z}_j + (1 - i)D_j, \quad E = \frac{1}{4\pi\lambda}\int_L \{\omega(t)[\lambda t^{\cdot} - t\theta_2(t)] + \overline{\omega(t)}[\lambda t^{\cdot} - t\theta_1(t)]\}\frac{\overline{dt}}{\bar{t}}$$

4. We now consider the solvability of the integral equation (3.1). First we establish an important property of equation (3.1). We integrate the limiting condition (1.3) with its associated operator $\Gamma[\omega(t), t_0]$ (the same as the real part of condition (1.7)) term by term along the arc of the curve $L_j (j = 1, 2, \dots, m + 1)$. Since the left-hand side of formula (1.3) exclusive of this operator appears as a derivative with repsect to the arc of the single-valued function $w(x, y)$ in the enclosed region S, we shall have

$$\operatorname{Re}\varphi'_{m+1}(0) = 0 \quad \text{on } L_{m+1}, \qquad (w + w_0)_{t=a_{m+1}} - (w + w_0)_{t=a_j} = 0 \quad \text{on } L_j (j = 1, \dots, m) \tag{4.1}$$

Any solution of the integral equation (3.1) necessarily satisfies (4.1); in other words, the transformed conditions (1.3) and (1.4) remain equivalent to their original form thanks to the operator introduced, but now meet the requirements of a problem which demands strict compliance with the relations in (4.1).

Note 1. A clarification of the essential significance of (4.1) is easy. In place of the given boundary values of the function $w(x, y)$ as usually written, we take values of its derivative along the arc L in accordance with (1.3). A similar substitution for one of the conditions of the problem leads to a solution in a finite sum satisfying the boundary values of $w(x, y)$ with an accuracy up to a certain constant on each of the curves $L_j (j = 1, \dots, m + 1)$; these constants, generally speaking, differ from one another on different sections of the curves $L_j (j = 1, \dots, m + 1)$;

this greatly complicates the process of arriving at a solution to the point of completion - a process arising out of layers of operations both cumbersome and difficult. The very solution itself is as a rule not always acceptable for use. But thanks to the relations (4.1), particularly those between the constants, the complications disappear. The function $w(x, y)$ which may be found in a given case will differ from the function sought by a constant on the various L_j curves which make up the complete boundary L. Consequently the function will differ in the region S from the function sought by the same constant. The solution itself will be free from the exceptions found in other work, which sometimes bristles with formidable difficulties. (In order to avoid misunderstanding, we feel it necessary to make the reservation that we are fundamentally thinking of the calculational aspect of the problem.)

We now suppose that the homogeneous (for $f(t) = 0$) integral equation (3.1) has a certain nontrivial solution $\omega_0(t)$. Corresponding to this value of density, the functions (2.1), (2.2), (2.3), (2.9) and (2.10), as well as those entering into the functionals A_j, B_j and D_j, are given zero subscripts. We introduce $\phi_{0j}(z)$, $\psi_{0j}(z)$, $\phi_{0j}^{(j_0)}(z)$, $\psi_{0j}^{(j_0)}(z)$, $\phi_{0j}^*(z)$ and $\psi_{0j}^*(z)$, expressible in sums like (1.5), corresponding to the functions $\phi_0(z)$, ..., $\psi_0^*(z)$; each of the last two pairs is determined in the corresponding field of the integrals (2.3) and (2.4) situated along the curve L_k and supplemented by additional terms for $j = k$. Finally we write the zero subscript on the biharmonic functions in (2.16) and (2.17).

The biharmonic function $w_0(x, y)$ mentioned previously in connection with (4.1) takes on the same constant value on all curves $L_j (j = 1, \ldots, m + 1)$ and also satisfies the homogeneous condition (1.7). It is not difficult to establish that $w_0(x, y) =$ constant everywhere in the region S on the basis of the integral formula for biharmonic functions [1]

$$\int_L \left[wH(w) - \frac{dw}{dn}G(w)\right] ds =$$

$$= -\iint_S \{\nu(\Delta w)^2 + (1-\nu)[(w_{xx})^2 + (w_{yy})^2 + 2(w_{xy})^2]\} dx dy$$

where ν is Poisson's ratio, $G(w)$ coincides with the left-hand side of (1.4) up to a constant multiplier, and where

$$H(w) = \frac{d\Delta w}{dn} + (1-\nu)\frac{d}{ds}[w_{xy}\cos 2\vartheta + (w_{yy} - w_{xx})\cos\vartheta \sin\vartheta]$$

From this we arrive at the conclusion that

$$\varphi_0(z) = kz + C, \qquad \psi_0(z) = -C \tag{4.2}$$

where k is a certain real constant and C a complex constant. From the last equations we get at once that $A_{0j} = B_{0j} = 0$ $(j = 1, \ldots, m)$. Consequently, the functions $\phi_0(z)$, $\psi_0(z)$, and at the same time the functions

$\phi_0^{(0)}(z)$, $\psi_0^{(0)}(z)$, $\phi_0^*(z)$ and $\psi_0^*(z)$, are regular in the region S. By taking this into account and operating on the principle of analytic continuation we obtain

$$\varphi_{0,m+1}(z) = kz + C, \qquad \psi_{0,m+1}(z) = -\bar{C}, \qquad \varphi_{0j}(z) = \psi_{0j}(z) = 0 \quad (j = 1, \ldots, m) \tag{4.3}$$

of which the first two equations hold good inside L_{m+1} and the other two outside L_j. We find, upon returning to (2.1)-(2.4), (2.12) and to the first equation of (4.1), that

$$k = C = 0, \quad \frac{1}{4\pi\lambda}\int_{L_{m+1}} \{\omega(t)\theta_1(t) + \overline{\omega(t)}\theta_2(t)\}\, dt = 0 \tag{4.4}$$

By letting $z \to \infty$ in formula (2.10) relating $\psi_{0j}(z)$ to $\psi_{0j}^*(z)$, we find that $D_{0j} = 0$ ($j = 1, \ldots, m$). Thus, in the equations for the functionals there have been evaluated

$$A_{0j} = B_{0j} = D_{0j} = 0 \quad (j = 1, \ldots, m) \tag{4.5}$$

Note 2. By virtue of these relations, $w_0(x, y)$ and $w_0^*(x, y)$ coincide, and the functions $w_{0j}(x, y) = w_{0j}^*(x, y)$ vanish at infinity as R^{-1} ($R = \sqrt{x^2 + y^2}$). Therefore the curvilinear integral along the contour L_j of the harmonic generating function Λw_{0j} reduces to zero.

We introduce the functions $\kappa(t)$ and $\mu(t)$ on the contour L, which take on the following values on each of the curves L_j ($j = 1, \ldots, m + 1$):

$$\varkappa(t) = \varkappa_j(t), \quad \varkappa_j(t) = \varkappa_j^*(t) - 2i\lambda\varepsilon_j(\bar{B} - E), \qquad \varkappa_j(a_j) = -2i\lambda\varepsilon_j(\bar{B} - E)$$

$$\varkappa_j^*(t) = \int_{a_j}^{t} [\omega(t)\theta_1(t) + \overline{\omega(t)}\theta_2(t)]\, dt, \qquad \varkappa_j^*(a_j) = 0 \tag{4.6}$$

$$\mu(t) = \mu_j(t); \quad -\mu_j(t) = -\overline{\varkappa_j(t)} + \{[\lambda \bar{t}' - \bar{t}\,\overline{\theta_1(t)}]\,\omega_0(t) + [\lambda \bar{t}' - \bar{t}\,\overline{\theta_2(t)}]\,\overline{\omega_0(t)}\}$$

while the functionals have values

$$A = \frac{1}{4\pi\lambda}\int_{L_{m+1}} \frac{\varkappa_{m+1}^*(t)}{t}\, dt, \qquad B = \frac{1}{4\pi\lambda}\int_{L_{m+1}} \frac{\overline{\varkappa_{m+1}^*(t)}\, dt}{t}$$

It is at once apparent that as a result of the last equation in (4.4) or the second in (4.5), the function $\kappa_j(t)$ is single-valued on the curve L_j. With these equations in mind, expressions (2.1) and (2.2) are obtained in the form

$$\varphi_0(z) = -\frac{1}{4\pi\lambda}\int_L \frac{\varkappa(t) + 2i\lambda\varepsilon_j(A + \bar{B} - E)}{t - z}\, dt, \quad \psi_0(z) = -\frac{1}{4\pi\lambda}\int_L \frac{\mu(t)}{t - z}\, dt \tag{4.7}$$

by employing simple transformations and integration by parts. (The point z lies in S.)

It is clear from (4.3) and the first two equations of (4.4) that each of the functions $\kappa_j(t)$ and $\mu_j(t)$ is analytically continuous and regular in a corresponding simply-connected region S bounded by L_j ($j = 1, \ldots, m+1$); of these, $\kappa_{m+1}(z) + 2i\lambda(A + B - E)$ and $\mu_{m+1}(z)$ vanish at infinitely remote parts of the region S_{m+1}. We differentiate the first of relations (4.6) with respect to t and solve for $\omega_0(t)$, thereby obtaining

$$\omega_0(t) = \frac{1}{4it\bar{t}'}\{\overline{\theta_1(t)}\,\varkappa_j'(t) - \theta_2(t)\,\overline{\varkappa_j'(t)}\} \qquad \text{on } L_j\ (j = 1, \ldots, m+1) \quad (4.8)$$

In this equation we set $\omega_0(t) = \nu_1^{(0)}(S) + i\nu_2^{(0)}(S)$ and separate into real and imaginary parts, with the result that

$$\nu_1^{(0)}(s) = \frac{1}{4\bar{t}t'}[\varkappa_j'(t) - \overline{\varkappa_j'(t)}], \qquad \nu_2^{(0)}(s) = -\frac{1}{4}[\varkappa_j'(t) + \overline{\varkappa_j'(t)}] \quad (4.9)$$

By introduction of this value of $\omega_0(t)$ into the second formula of (4.6) we find, after elementary calculations,

$$\varkappa_j(t) - t\overline{\varkappa_j'(t)} - \overline{\mu_j(t)} = \frac{\lambda t'}{2\bar{t}t'}[\varkappa_j'(t) - \overline{\varkappa_j'(t)}] \quad (4.10)$$

Since the functions $\kappa_j(z)$ and $\mu_j(z)$ are continuous on the curve L_j (taken as sufficiently smooth), the right-hand side of (4.1), must be so too, in spite of the possibility of the denominator reducing to zero at certain points on L_j. Multiply (4.10) by the quantity $t\,\theta_1(t)$ and its conjugate by $t\,\theta_2(t')$, and subtract one from the other term by term. As a result we obtain

$$\lambda[\varkappa_j'(t) - \overline{\varkappa_j'(t)}] + \bar{t}'\overline{\theta_1(t)}\gamma_j(t) - t'\theta_2(t)\overline{\gamma_j(t)} = 0 \quad (4.11)$$

where by analogy with (1.8)

$$\gamma_j(t) = \varkappa_j(t) - t\overline{\varkappa_j'(t)} - \overline{\mu_j(t)}$$

Formula (4.11) coincides essentially with the homogeneous boundary condition (1.7) on the curve L_j. In accordance with the Goursat formula we construct the biharmonic function

$$2\sigma_j(x, y) = -i\{\bar{z}\varkappa_j(z) - z\overline{\varkappa_j(z)} + \tau_j(z) - \overline{\tau_j(z)}\}, \quad \tau_j(z) = \int^z \mu_j(z)\,dz \quad (4.12)$$

in the region S_j. It evidently satisfies the conditions

$$\frac{\partial\sigma_j}{\partial s} = 0, \qquad G(\sigma_j) = 0 \qquad \text{on } L_j\ (j = 1, \ldots, m+1) \quad (4.13)$$

It is not difficult to imagine that at an infinitely remote part of the region S_{m+1} the behavior of the function $\sigma_{m+1}(x, y)$ is described by the relation

$$\sigma_{m+1}(x, y) = \overline{E}_{m+1}z + E_{m+1}\bar{z} + D_{m+1}\ln R + \sigma_{m+1}^{*}(x, y) \qquad (4.14)$$

where $\sigma_{m+1}^{*}(x, y)$ is a limiting biharmonic function, D_{m+1} and E_{m+1} are certain real and complex constants (the function $\tau_{m+1}(z)$ does not contain a term of the form D_{m+1} ln z, else it would acquire an increment during its circuit of the contour L_{m+1} and would not maintain its constant value). For the given conditions it is easy to establish that $\sigma_{m+1}(x, y)$ must be identically equal to a constant everywhere outside L_{m+1}; this requires the equality $\kappa_{m+1}^{*}(z) = \mu_{m+1}(z) = 0$; it follows also that $A = B = 0$.

By similar reasoning, recalling that $\kappa_j(a_j) = 0$, we find that the equations

$$\varkappa_j(z) = k_j(z - a_j), \qquad \mu_j(z) = 0 \qquad (j = 1, \ldots, m) \qquad (4.15)$$

hold good in the region S_j, where the k_j are certain real constants; from the first of equations (4.5) we see that these constants must be zero. Thus, all the functions

$$\varkappa_j(z) = \mu_j(z) = 0 \qquad (j = 1, \ldots, m + 1) \qquad (4.16)$$

By consideration along with (4.9) of a relation obtained from the second of the formulas in (4.6) and from (4.16), we find that

$$(\lambda\bar{t}' - \bar{t}t'\bar{t}'')\,\nu_1^{(0)}(s) - \bar{t}\nu_2^{(0)}(s) = 0 \qquad (4.17)$$

and at once conclude that $\omega_0(t)$ is necessarily zero.

Thus, the homogeneous equation (3.1) is always uniquely solvable. Upon determination of the density $\omega(t)$, we find the functions $\phi(z)$ and $\psi(z)$ from formulas (2.1) and (2.2). It is possible that the $w(x, y)$ obtained from them will differ from the function required by a certain constant value. By making the appropriate correction the required solution to the problem is obtained.

Note 3. We remark that in equation (4.17) we necessarily resort to appropriate junctions (for smoothness) with contiguous arcs, should any of the curves L_j consist of rectilinear parts.

Note 4. The removal of $iD_{0j}/z - z_j$ from equation (2.2) leads to the conclusion that in this case D_{0j} will, generally speaking, be different from zero as a result of the vanishing of the functional terms in $\psi_{0j}(z)$; evidently, instead of $\psi_{0j}^{0}(z) = 0$, we have the relation

$$\psi_{0j}^{(0)}(z) + D_{0j} / z - z_j = 0$$

the reasoning and the conclusions then lose force in certain essential particulars. Indeed, the Fredholm equation will not be solvable with this change in viewpoint. The attraction of the operator $iD_j/z - z_j$, as we have seen, is that it corrects the position in the desired direction.

Note 5. The problem considered here belongs to one of the most difficult branches of the theory of elasticity. There is therefore a tendency in this category of research towards the greatest generality, along with improvement and simplification of the methods of investigation. In the articles referred to [1, 2], the methods employed were different, at least in the first stages of the investigation, for a wide variety of problems in potential theory and elasticity theory. It is a pity that the application of such methods does not always lead to relatively simple results. In reference [3], simple expressions were taken for the unknown functions. Nevertheless, not being especially adapted to the problem in the sense as used here, they led the author to a system of singular integral equations belonging to a class for which satisfactory methods have not been developed. In spite of all this, it is apparent that the value of these researches has by no means been exhausted. Even now they all possess interest, each with its own point of view; and there is no doubt but that their significance and that of the present work will grow with the development of effective methods of solving integral equations.

The following interesting circumstance deserves mention, among other things. One may study certain particular problems by the integro-differential equations of references 1 and 2 by using elementary kernels (up to their transformation into Fredholm equations) and by searching for the unknown density in the form of a complex Fourier series; this leads to an infinite system of linear equations. This remark, in all probability, also applies to [3].

We are far from thinking that the solution to the problem in the present paper is the most simple of all possible solutions, including those which may be obtained by a reduction to a Fredholm equation or by any other means. At the same time we are inclined to think that another Fredholm equation for the problem with enough distinguishing relations and useful features for it to be clearly preferable to (3.1) can hardly be constructed.

5. We dwell briefly on the case where the region S is infinite and bounded by a contour L consisting of a totality of curves $L_j (j = 1, \ldots, m)$. In this case it is preferable to pass over (2.1) and (2.2) and to start with (2.9) and (2.10) as a basis. We assume that at an infinitely remote part of the region S the biharmonic function is

$$w(x, y) = Bz + \bar{B}\bar{z} + D \ln R + \ldots \tag{5.1}$$

where B is a complex and D a real constant, and where the multiple dots signify a limited summation in which the nth derivative is of order R^{-n}. In order for the behavior of $w(x, y)$ as described in (5.1) to be realized, it is necessary to set

$$A_k = -\sum_{j=1}^{m}{}' A_j, \qquad B_k = \sum_{j=1}^{m}{}' [iA_j(z_j - z_k) - B_j] \tag{5.2}$$

where k is a certain arbitrary fixed number from the series $j = 1, \ldots, m$ and where the prime on the summation symbol indicates a passage to the case $j = k$. In this connection, formulas (2.9) and (2.10) may take another form which displays their properties more prominently. Further, by including an additive function B_k (in agreement with (2.7)), and by taking expressions in the form of sums as in the second terms of (2.9), (2.10) and (2.11), we shall have

$$\varphi(z) = \varphi^*(z) + \sum_{j=1}^{m} [iA_j(z - z_j) + B_j] \ln \frac{z - z_j}{z - z_k} + B_k \tag{5.3}$$

$$\psi(z) = \psi^*(z) - \sum_{j=1}^{m} \left\{ (iA_j\bar{z}_j + B_j) \left[\ln \frac{z - z_j}{z - z_k} + \frac{z_j}{z - z_j} - \frac{z_k}{z - z_k} \right] + i \frac{D_j}{z - z_j} \right\} \tag{5.4}$$

The primes on the summation signs have been omitted in these equations since the summation automatically drops out for $j = k$. It is seen from (5.3) and (5.4) that as z goes to infinity the function $\phi(z)$ is bounded and $\psi(z)$ decreases according to the modulus of order z^{-1}. In formula (2.16) for $w(x, y)$ the values of δ_{1j}^* and δ_{2j}^* will depend upon the index k and are respectively equal to

$$-[|z - z_j|^2 - |z_j|^2] \ln \left| \frac{z - z_j}{z - z_k} \right|^2, \qquad i\bar{z} \ln \left| \frac{z - z_j}{z - z_k} \right|^2$$

The operator $\Gamma[\omega(t), t_0]$ must now be set equal to

$$\Gamma[\omega(t), t_0] = \{[w + w_0]_{t=a_j} - [w + w_0]_{t=a_k} \quad \text{on } L_j (j \neq k); \qquad A_k \quad \text{on } L_k\} \tag{5.5}$$

is distinction from (1.6), and in which the functional A_k is taken to be the same as in (2.1). It is not difficult to convince oneself that the integral formula preceding (4.2) holds good also for an infinite region S for the conditions of (5.2). The integral equation for the density $\omega(t)$ is somewhat different in this case from (3.1); it is easily written starting from (5.3) and (5.4). Just as in (3.1), it always has a unique solution. From the determination of $\omega(t)$, we find $w_1(x, y)$ acquiring a constant value on all $L_j (j = 1, \ldots, m)$. Subtraction of this constant value gives the required solution to the problem.

Note 1. The problem of the determination of the biharmonic function has, generally speaking, no solution for conditions at infinity more restricted than (5.1). In fact, let there be another solution to the problem for the case where B and D are not zero at the same time for any given $f(t)$. The biharmonic function consisting of the difference of these two solutions satisfies the homogeneous conditions (1.3) and (1.4) and for large $|z|$ has the same order as (5.1). We conclude from this that the function is identically equal to a constant; this signifies that the quantities and D must reduce to zero, which is impossible.

If $w(x, y)$ possesses at infinity a higher order than in (5.1), then additional terms with singularities higher than those in (5.1) must invariably be given.

Note 2. We assume that the normal load $q(x, y)$ acting on the bent plate is distributed over a certain area Ω, understood to be a finite part of the region S. In this case a particular solution of the differential equation of bonding which vanishes at infinity may be taken as

$$w_0(x,y) = \frac{1}{16\pi D}\int_\Omega q(\xi,\eta)\, r^2 \ln r\, d\Omega + \Delta_1^{(0)} r_0^2 \ln r_0 + (2\ln r_0 + 1)\,[\Delta_2^{(0)} + \Delta_1^{(1)}(x-x_0) + \Delta_2^{(1)}(y-y_0)] + \Delta_{1,1}^{(2)}\left(\frac{x-x_0}{r_0}\right)^2 + \Delta_{1,2}^{(2)}\frac{(x-x_0)(y-y_0)}{r_0^2} + \Delta_{2,2}^{(2)}\left(\frac{y-y_0}{r_0}\right)^2$$

where D is the cylindrical stiffness. Here the point $M(x_0, y_0)$ lies outside the region S, and

$$\Delta_1^{(0)} = -\frac{1}{16\pi D}\int_\Omega q(\xi,\eta)\, d\Omega, \qquad \Delta_2^{(0)} = -\frac{1}{32\pi D}\int_\Omega q(\xi,\eta)\,\rho_0^2(\xi,\eta)\, d\Omega$$

$$\Delta_1^{(1)} = \frac{1}{16\pi D}\int_\Omega q(\xi,\eta)(\xi-x_0)\, d\Omega, \qquad \Delta_2^{(1)} = \frac{1}{16\pi D}\int_\Omega q(\xi,\eta)(\eta-y_0)\, d\Omega$$

$$\Delta_{1,1}^{(2)} = -\frac{1}{16\pi D}\int_\Omega q(\xi,\eta)(\xi-x_0)^2\, d\Omega, \qquad r_0 = \sqrt{(x-x_0)^2+(y-y_0)^2}$$

$$\Delta_{1,2}^{(2)} = -\frac{1}{8\pi D}\int_\Omega q(\xi,\eta)(\xi-x_0)(\eta-y_0)\, d\Omega, \qquad \rho_0 = \sqrt{(\xi-x_0)^2+(\eta-y_0)^2}$$

$$\Delta_{2,2}^{(2)} = -\frac{1}{16\pi D}\int_\Omega q(\xi,\eta)(\eta-y_0)^2\, d\Omega$$

Note 3. A suitable method, well tested in its applications to a wide range of problems in elasticity theory [6], may be demonstrated for a certain shape of the region S. For example, let the region S be an eccentric ring bounded by circles L_j of radius R_j, and with centers at z_j $(j = 1, 2)$. The boundary condition (1.7) for the functions $\phi^*(z)$ and $\psi^*(z)$ for this case, to be introduced into formulas (2.9) and (2.10), is written as

$$\lambda[\varphi^{*\prime}(t)-\varphi^{*\prime}(t)]-(1+R_j^{-1})\frac{R_j}{t-z_j}\gamma^*(t)-(1-R_j^{-1})\frac{t-z_j}{R_j}\overline{\gamma^*(t)}=f^*(t) \tag{5.6}$$

where the sense of $\gamma^*(t)$ is apparent; quantities containing multiplying parameters A_1, B_1 and D_1 are related to the function $f^*(t)$.

Proceeding as in reference [6], we introduce an auxiliary function $\omega^*(t)$ on the internal circumference L specified by the relation

$$\lambda[\varphi^{*\prime}(t)+\overline{\varphi^{*\prime}(t)}]-(1+R_1^{-1})\frac{R_1}{t-z_1}\eta^*(t)+(1-R_1^{-1})\frac{t-z_1}{R_1}\overline{\eta^*(t)}=2\omega^*(t) \tag{5.7}$$

in which

$$\eta^*(t)=\varphi^*(t)+t\overline{\varphi^{*\prime}(t)}+\overline{\psi^*(t)}$$

By successively adding and subtracting (5.6) and (5.7), term by term, the following relation

$$\mp(1\pm R_1^{-1})\frac{R_1}{t-z_1}\varphi^*(t)+\left\{\lambda\pm(1\mp R_1^{-1})\left[R_1+\bar{z}_1\frac{t-z_1}{R_1}\right]\right\}\varphi^{*\prime}(t)$$
$$\pm(1\mp R_1^{-1})\frac{t-z_1}{R_1}\psi^*(t)=\omega^*(t)\pm\frac{1}{2}f^*(t)$$

is obtained, which holds simultaneously for the upper and the lower signs.

By excluding first $\phi^*(t)$ and then $\psi^*(t)$ from these, we are led to the formulas

$$-\frac{2}{t-z_1}\varphi^*(t)+\lambda\varphi^{*\prime}(t)=F^*(t)$$
$$\left[-2\frac{\bar{z}_1}{R_1}+(\lambda-2)\frac{R_1}{t-z_1}\right]\varphi^{*\prime}(t)-\frac{2}{R_1}\psi^*(t)=G^*(t)$$

in which the right-hand members are:

$$\left[F^*(t);\ \frac{t-z_1}{R_1}G^*(t)\right]=\frac{1}{2}\left\{(1\pm R_1^{-1})\left[\omega^*(t)+\frac{1}{2}f^*(t)\right]+(1\mp R_1^{-1})\left[\overline{\omega^*(t)}-\frac{1}{2}\overline{f^*(t)}\right]\right\}$$

Here the upper and lower signs appearing in the expression on the right relate respectively to the first and the second of the functions on the left. Each of the functions $\omega^*(t)$ and $f^*(t)$ can be written expediently in the form of a complex Fourier series, in powers of $t-a_1$. Further, we introduce the following functions which are regular in the region S:

$$\delta^*(z)=-\frac{2}{z-z_1}\varphi^*(z)+\lambda\varphi^{*\prime}(z)-\frac{1}{2\pi i}\int_{L_1}\frac{F^*(t)}{t-z}dt$$

$$\chi^*(z)=\left[-2\frac{\bar{z}_1}{R_1}+(\lambda-2)\frac{R_1}{z-z_1}\right]\varphi^{*\prime}(z)-\frac{2}{R_1}\psi^*(z)-\frac{1}{2\pi i}\int_{L_1}\frac{G^*(t)}{t-z}dt.$$

They will be analytically continuous and regular inside the circumference of L_2. Considering $\omega^*(t)$ as conditionally given, we find the functions $\delta^*(z)$ and $\chi^*(z)$ from the limiting equation on the external circumference. By then passing to the condition (5.7) on L_1, we obtain an infinite system of linear equations for the unknown coefficients in the expansion of $\omega^*(t)$; it is readily established that this system will be quasiregular for any region shape. By solving this system and so determining $\omega^*(t)$, we are then able to write the expression for the unknown functions.

Note 4. Fridman has drawn attention to the analogy between the above problem and the plane problem of the theory of elasticity for given boundary values of the normal components of the vector displacement and the tangential component of the stress vector. This problem was considered in reference [7] for the case of finite simply-connected regions. The limiting equation there presented may be transformed into

$$(\varkappa - 1)\,[\varphi'(t) - \overline{\varphi'(t)}] - t^{\cdot}\theta_2^{\cdot}(t)\,\overline{\gamma(t)} + \bar{t}^{\cdot}\overline{\theta_1(t)}\,\gamma(t) = f(t) \tag{5.8}$$

Here μ and κ are elastic constants, and

$$\gamma(t) = \varkappa\varphi(t) + \overline{t\varphi'(t)} + \overline{\psi(t)}, \qquad f(t) = -4\mu\left[v_n + i\left(\frac{dv_n}{ds} + T/2\mu\right)\right] \tag{5.9}$$

in which v_n is the normal component of the displacement vector and T is the tangential component of the stress vector. A comparison of these equations with (1.7) at once confirms the correctness of Fridman's observation.

The functions $\phi(z)$ and $\psi(z)$ for a finite multiply-connected region S may be taken in the form

$$\begin{aligned} \varphi(z) &= \int_L \{\omega(t)\,\theta_1(t) + \overline{\omega(t)}\theta_2(t)\}\, G(t, z)\, dt \\ \psi(z) &= \int_L \{\omega(t)\, H(t, z) + \overline{\omega(t)}\, T(t, z)\} dt \end{aligned} \tag{5.10}$$

following the above analogy, in which, without going into detail, neither of the curves L_j is considered to be a circle.

The functions $G(t, z)$, $H(t, z)$ and $T(t, z)$ are more general than in (2.5) and (2.6) and are given by the formulas

$$G(t, z) = \left\{-\frac{1}{4\pi(\varkappa - 1)}\left[-1 + \ln\left(1 - \frac{z}{t}\right)\right],\; -\frac{1}{4\pi(\varkappa - 1)}\ln(z - t)\right\}$$

$$H(t, z) = \varkappa\bar{t}^{\cdot 2}\overline{\theta_2(t)}\left[-G(t, z) + \varepsilon_j\,\frac{1}{4\pi(\varkappa - 1)}\right] + P(t)\left[\frac{1}{t - z} - \varepsilon_j\,\frac{1}{t}\right]$$

$$T(t, z) = \varkappa \bar{t}^{\prime 2} \overline{\theta_1(t)} \left[-G(t, z) + \varepsilon_j \frac{1}{4\pi(\varkappa - 1)} \right] + Q(t) \left[\frac{1}{t - z} - \varepsilon_j \frac{1}{t} \right]$$

$$P(t) = \frac{1}{4\pi(\varkappa - 1)} [(\varkappa - 1)\bar{t}' + \bar{t}\overline{\theta_1(t)}], \qquad Q(t) = \frac{1}{4\pi(\varkappa - 1)} [(\varkappa - 1)\bar{t}' + \bar{t}\overline{\theta_2(t)}]$$

in which ϵ_j means the same as before. By the introduction of (5.10) into the boundary condition (5.8), we obtain a Fredholm integral equation which is uniquely solvable for the density $\omega(t)$.

When in this case the multiply-connected region S is infinite, it is natural to consider the principal vector of the external forces acting on the region bounded by the curves $L_j (j = 1, \ldots, m)$ as known. We set

$$\varphi(z) = B \ln(z - z_k) + \varphi_0(z), \qquad \psi(z) = -\varkappa \bar{B} \ln(z - z_k) + \psi_0(z) \qquad (5.11)$$

in which the functions $\phi_0(z)$ and $\psi_0(z)$, single-valued in the region S, are equal to

$$\varphi_0(z) = \int_L \{\omega(t)\theta_1(t) + \overline{\omega(t)}\theta_2(t)\} G(t, z)\, dt - \left(\sum_{j=1}^{m} B_j\right) \ln(z - z_k) + B_k \qquad (5.12)$$

$$\psi_0(z) = \int_L \{\omega(t) H(t, z) + \overline{\omega(t)}\, T(t, z)\}\, dt + \varkappa \left(\sum_{j=1}^{m} \bar{B}_j\right) \ln(z - z_k)$$

In these formulas the index k is a certain number from the series $j = 1, \ldots, m$ and the constant B is

$$B = \frac{1}{2\pi i(1 + \varkappa)} (X + iY)$$

where $X + iY$ is the principal stress vector at an infinitely remote part of the plane. We may convince ourselves that $\phi_0(z) = B_k$ and that $\psi_0(z) = 0$ by considering a homogeneous system of equations containing quantities with the constant B as a multiplier, as well as free terms. We find that $B_k = 0$ from the condition that the normal component of the displacement reduces to zero on the contour L, and, also that $\phi_0(z) = 0$. The rest of the proof of solvability follows just as above.

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